

CONTRACTIBLE POLYHEDRA IN PRODUCTS OF TREES AND ABSOLUTE RETRACTS IN PRODUCTS OF DENDRITES

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ABSTRACT. We show that a compact n -polyhedron PL embeds in a product of n trees if and only if it collapses onto an $(n-1)$ -polyhedron. If the n -polyhedron is contractible and $n \neq 3$ (or $n = 3$ and the Andrews–Curtis Conjecture holds), the product of trees may be assumed to collapse onto the image of the embedding.

In contrast, there exists a 2-dimensional compact absolute retract X such that $X \times I^k$ does not embed in any product of $2+k$ dendrites for each k .

1. INTRODUCTION

All *spaces* shall be assumed to be metrizable. By a *compactum* we mean a compact metrizable space. A finite-dimensional compactum is an *ANR* if and only if it is locally contractible; and an *absolute retract (AR)* if and only if it is a contractible ANR (see [5]). A one-dimensional compact AR is called a *dendrite*, and a one-dimensional compact ANR is called a *local dendrite*. An arbitrary connected one-dimensional compactum is sometimes called a *curve*.

Theorem 1.1 (Nagata–Bowers [39], [7]; see also [46], [47], [2]). *Every n -dimensional compactum X embeds in $D^n \times I$, where D is a certain dendrite.*

It is well-known that every dendrite embeds in the 2-cube I^2 ; thus Theorem 1.1 may be viewed as an improvement of the classical Menger–Nöbeling–Pontriagin theorem that every n -dimensional compactum embeds in the $(2n+1)$ -cube I^{2n+1} .

Theorem 1.1 is trivial in the case where X is a polyhedron:

Theorem 1.2. *Every compact n -polyhedron embeds in a product of n trees and I .*

Proof. Given a triangulation K of the given polyhedron X , let S_i be the set of all vertices of the barycentric subdivision K' that are barycenters of i -simplices of K . The simplicial map $K' \rightarrow S_0 * \dots * S_n$ is clearly an embedding. Hence X embeds in $I * S$, where $S = S_1 * \dots * S_n$. Next, $I * S = pt * CS$ is homeomorphic to $pt * (CS \cup S \times I) = I \times CS$. Finally, the cone CS is homeomorphic to the product of n trees $CS_1 \times \dots \times CS_n$. \square

The above argument yields an explicit embedding of every compact n -polyhedron in I^{2n+1} , which we have not seen in the literature. This is strange, for a part of this construction is certainly well-known; it yields

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Proposition 1.3 ([17], [24]). *The cone over every compact n -polyhedron embeds in a product of $n + 1$ trees.*

Theorem 1.4 (Borsuk–Patkowska [6]). *The n -sphere S^n does not embed in any product of n dendrites, for each $n \geq 0$.*

Another proof of Theorem 1.4 is given by the easy part of our Theorem 1.9 below.

Theorem 1.5 (Gillman–Matveev–Rølfesen [17]). *Every compact connected PL n -manifold with nonempty boundary embeds in a product of n trees.*

This was originally a consequence of Proposition 1.3 along with a “reconstruction theorem” announced in [17]. Another proof of Theorem 1.5 is given by our Theorem 1.9 below, albeit the trees that it produces need not be cones over finite sets.

Nagata’s original motivation for considering embedding into products of 1-dimensional spaces related to dimension theory (see [39]). Borsuk’s proof of the 2-dimensional case of Theorem 1.4 was a solution to Nagata’s problem; on the other hand, the author learned from W. Kuperberg, a student of Borsuk who has generalized Theorem 1.4 [29], that Borsuk saw this result as a part of his work on the problem of uniqueness of decomposition of ANRs into products. Yet another motivation for embedding into products of trees was the Poincaré Conjecture (now also known as Perelman’s Theorem):

Theorem 1.6. (a) (Gillman [16]) *If a compact acyclic 3-manifold embeds in the product of a tree and I^2 , then it is collapsible.*

(b) (Zhongmou [54]) *Every compact connected 3-manifold with nonempty boundary embeds in the product of two triods and I .*

A discussion of further results in the theory of embeddings into products of dendrites (or curves) can be found in the recent paper [24], which itself is a significant addition to this theory (see also additional details in [25]). We should mention

Theorem 1.7 (Koyama–Krasinkiewicz–Spież [24]). *There exists a 2-polyhedron that collapses onto a product of two graphs but does not embed in any product of two graphs. Yet it embeds in a product of two curves.*

The 2-polyhedron in question is $\Theta \times \Theta \bigcup_{J=I \times \{0\}} I \times I$, where Θ is the suspension over the three-point set, and the arc J lies in the interior of a 2-cell of $\Theta \times \Theta$ apart from one endpoint, which lies in a “corner” of that 2-cell.

1.A. Embedding contractible polyhedra in products of trees

Theorem 1.8. *Every collapsible compact n -polyhedron PL embeds in a product of n trees. Moreover, the product of trees collapses onto the image of the embedding.*

The embeddability in the 2-dimensional case is due to Koyama, Krasinkiewicz and Spież [24]. The principal additional ingredient in our proof of the general case is the Fisk–Izmestiev–Witte lemma [15; Lemma 57], [22], [52] (see also [19; Lemma 3.1], [10]),

which asserts that for every finite set C (the ‘palette’) of cardinality $\#C \geq d + 1$, every C -colored combinatorial $(d - 1)$ -sphere is color-preserving isomorphic to the boundary of a C -colored combinatorial ball. (A simplicial complex is C -colored if its vertices are colored by the elements of C so that no edge connects two vertices of the same color.)

In particular, this lemma implies that if a triangulation of S^2 admits a 4-coloring, then it extends to a triangulation of the 3-ball where the link of every interior edge is (combinatorially) an even-sided polygon. As observed by R. D. Edwards and others in 1970s, the converse to this also holds: every such triangulation of the 3-ball has a 4-colorable boundary (see references in [22]).

The 2-dimensional case of Theorem 1.8 involves only the trivial case $d \leq 1$ of the Fisk–Izmestiev–Witte lemma.

Corollary 1.9. *Let P be a compact n -polyhedron. The following are equivalent:*

- (i) P PL embeds in a product of n trees;
- (ii) P PL embeds in a product of an $(n - 1)$ -polyhedron and a tree;
- (iii) P collapses onto an $(n - 1)$ -polyhedron;
- (iv) P PL embeds in a collapsible compact n -polyhedron.

Here (iv) \Rightarrow (i) follows from Theorem 1.8, (i) \Rightarrow (ii) is obvious, (ii) \Rightarrow (iii) is easy (see below), and to see that (iii) \Rightarrow (iv) it suffices to note that if P collapses onto Q then the amalgamated union $P \cup_Q CQ$ is collapsible, where CQ is the cone over Q .

Alternatively, (i) \Rightarrow (iv) is obvious, and another proof of (iii) \Rightarrow (i) is given in §2.

Proof of (ii) \Rightarrow (iii). Suppose that P is embedded in $R \times T$, where R is an $(n - 1)$ -polyhedron and T is a tree, and P does not collapse onto any $(n - 1)$ -polyhedron. Let P_0 be a triangulation of P . Then P_0 collapses onto a (generally non-unique) simplicial complex Q_0 that does not collapse onto any proper subcomplex. Then Q_0 has no free faces, and it follows that $Q := |Q_0|$ does not collapse onto any proper subpolyhedron. By the hypothesis Q is of dimension precisely n . The projection $f: Q \subset R \times T \rightarrow R$ can be triangulated by a simplicial map $Q_1 \rightarrow R_1$. Let p is a point in the interior U of a top-dimensional simplex of R_1 such that the corresponding fiber $F := f^{-1}(p)$ is of dimension precisely 1. The projection $F \subset R \times T \rightarrow T$ is an embedding, so F is a forest. Thus F collapses onto a finite set, but is not a finite set itself; so it must have a free vertex. On the other hand, $f^{-1}(U) \cong F \times U$ by Pontryagin’s lemma [40; Proposition C], [51; Theorem 1.3.1] (see also [11; §5]). Hence Q_1 has a free face. Thus Q collapses onto a proper subpolyhedron, which is a contradiction. \square

Corollary 1.10 (Koyama–Krasinkiewicz–Spieź [24]). *An acyclic compact 2-polyhedron P embeds in a product of two trees if and only if P is collapsible.*

Remark 1.11. Let P be a compact polyhedron with $H^1(P) = 0$. If P embeds in a product of n graphs then it embeds in a product of n trees, namely in the product of (appropriate compact subtrees of) the universal covers of the n graphs. Thus “trees” can be replaced with “graphs” in Corollary 1.10 in accordance with [24]. (In fact, it was shown in [24]

that an acyclic non-collapsible compact 2-polyhedron does not embed in any product of two curves.)

Corollary 1.12. *Let P be an n -polyhedron. For $n \neq 3$, the following are equivalent:*

- (i) *some product of n trees collapses onto a PL copy of P ;*
- (ii) *P collapses onto a contractible $(n - 1)$ -polyhedron;*
- (iii) *some collapsible compact n -polyhedron collapses onto a PL copy of P .*

For $n = 3$, the same holds with “contractible” replaced by “3-deformable to a point”.

A polyhedron P is said to be n -deformable to a polyhedron Q if they are related by a sequence of collapses and expansions (i.e. the inverses of collapses) through polyhedra of dimensions $\leq n$. The Andrews–Curtis Conjecture asserts that all contractible 2-polyhedra 3-deform to a point (see [1], [33]). Among its motivations (cf. Curtis [13; §2]) we mention that it would imply¹ that every contractible 2-polyhedron PL embeds in I^4 .

Proof. (iii) \Rightarrow (i) follows from Theorem 1.8 and (i) \Rightarrow (ii) follows from Corollary 1.9. To prove (ii) \Rightarrow (iii), suppose that P collapses onto an $(n - 1)$ -polyhedron Q , and either Q is contractible, or $n = 3$ and Q 3-deforms to a point. Then by a result of Kreher–Metzler and Wall, there exists an $(n - 1)$ -polyhedron R such that R collapses onto a PL copy of Q and $R \times I$ is collapsible [28; Satz 1a, Satz 1] (see also [1; §XI.4] for an outline of Kreher and Metzler’s proof in English). Let S be the amalgamated union $P \cup_{Q=Q \times \{0\}} R \times I$. Then $S \searrow R \times I \searrow pt$ and $S \searrow P \cup_Q R \searrow P$. \square

Remark 1.13. For each $n \geq 3$ it is easy to construct a non-collapsible n -polyhedron that collapses onto a contractible $(n - 1)$ -polyhedron (e.g. $I^n \vee \text{cone}(f)$ will do, where f is any degree 0 PL surjection $S^{n-2} \rightarrow S^{n-2}$). A more interesting example is due to M. M. Cohen, who constructed for each $n \geq 4$ a contractible $(n - 1)$ -polyhedron Q such that $Q \times I$ is not collapsible [12]. Other constructions (with very different proofs) are now known: $P \times I^{k-2}$ is not collapsible if P is the suspension over a $(k - 1)$ -dimensional spine of a non-simply-connected homology k -sphere [3], and $P \times I^q$ is not collapsible if P is a certain “ $(3q + 6)$ -dimensional dunce hat” [8].

A *free deformation retraction* of a space X onto a subspace Y is a homotopy $h_t: X \rightarrow X$ starting with $h_0 = \text{id}$, ending with a retraction h_1 of X onto Y , and such that $h_t h_s = h_{\max(s,t)}$ for all $s, t \in [0, 1]$. A space is *freely contractible* if it freely deformation retracts onto a point. Collapsibility is known to be strictly stronger than topological collapsibility [3], [8] and consequently than free contractibility; however, in the case of 2-polyhedra the three notions are equivalent [21].

Conjecture 1.14. *A compact n -polyhedron collapses onto an $(n - 1)$ -polyhedron if and only if it freely deformation retracts onto an $(n - 1)$ -polyhedron.*

¹By general position every 2-polyhedron P immerses in I^4 , and therefore embeds in a 4-manifold M . Let N be a regular neighborhood of P in M . If P 3-deforms to a point, then the double of N is the 4-sphere (see [1; Assertion (59) in Ch. I]).

Remark 1.15. The proof of Theorem 1.8 involves a (non-straightforward) construction of a collapsible cubulation of the given collapsible polyhedron, which might be of interest in its own right. Another such construction (a more straightforward one) has been used to characterize collapsible polyhedra in the language of abstract convexity theory [48], and to establish the ‘only if’ part of Isbell’s conjecture: a compact polyhedron is collapsible if and only if it is injectively metrizable [30], [49; Chapter VI]. (Isbell himself proved that the two conditions are equivalent for 2-polyhedra [21].)

1.B. Embedding absolute retracts in products of dendrites

A map $f: X \rightarrow Y$ is called an ε -map with respect to some metric on X if every its point-inverse $f^{-1}(pt)$ is of diameter at most ε . A compactum X is said to *quasi-embed* in a space Y if for some (or equivalently, every) metric on X , it admits an ε -map into Y for each $\varepsilon > 0$. We refer to [42] for a definitive discussion of the (quite subtle) difference between embeddability and quasi-embeddability of compact polyhedra in I^m .

Our paper was originally motivated by the following problem.

Problem 1.16 (Koyama, Krasinkiewicz, Spieź [25]). *Suppose that X is a compactum, quasi-embeddable in the n th power of the Menger curve. Can X be embedded there?*

This problem appears as Problem 1.4 in [25] with the following comments: “Our next problem is of great interest, we believe it has affirmative solution.”

In the present paper, we shall prove

Theorem 1.17. *There exists a 2-dimensional compact AR X such that $X \times I^k$ quasi-embeds in a product of $2+k$ dendrites but does not embed in any product of $2+k$ curves, for each $k \geq 0$.*

The proof of the higher-dimensional (i.e. $k \geq 1$) case is similar to (and only three lines longer than) the proof of the two-dimensional case. Similar arguments show that the Cartesian power X^k quasi-embeds in a product of $2k$ dendrites, but does not embed in any product of $2k$ curves.

Remark 1.18. A few months after we shared our proof of the two-dimensional case of Theorem 1.17 with J. Krasinkiewicz and S. Spieź, they found their own solution of Problem 1.16 [27]. Compared to ours, it is amazingly simple (modulo their previous work with A. Koyama) — at least when slightly modified as follows.

The dunce hat D [53] (also known as the Borsuk tube [4], [27]) is easily seen to be the quotient of a collapsible polyhedron \hat{D} by its only free edge. Indeed, the link L of the 0-cell e_0 of D is homeomorphic to $S^1 \times \partial I \cup pt \times I$, where $pt \in S^1$ (cf. [53; Fig. 5]). Let $\pi: L \rightarrow I$ be the projection. The star S of e_0 in some triangulation of D is homeomorphic to the cone over L , which can be viewed as the mapping cylinder of the constant map $L \rightarrow pt$; we define \hat{D} by replacing S with the mapping cylinder $MC(\pi)$. The target space I of π is identified with a free edge J in \hat{D} , and clearly \hat{D} is collapsible.

The quotient map $\hat{D} \rightarrow \hat{D}/J = D$, being cell-like, is an ε -homotopy equivalence for each $\varepsilon > 0$ by Chernavsky’s lemma [23; Lemma 1]; in particular, for each $\varepsilon > 0$ there

exists an ε -map $f_\varepsilon: D \rightarrow \hat{D}$. (Specifically, f_ε is the identity outside S , and $f_\varepsilon|_S$ is the composition $S \xrightarrow{h} L \times I \bigcup_{L \times \{1\} = L * 0} L * pt \xrightarrow{g} MC(\pi)$, where h is a homeomorphism such that $h^{-1}(L * pt)$ lies in the $\frac{\varepsilon}{2}$ -neighborhood of e_0 , and g combines the quotient map $L \times I \rightarrow MC(\pi)$ with a null-homotopy $L * pt \rightarrow I$ of π .) Since \hat{D} is collapsible, it embeds in a product of two trees ([24]; see Corollary 1.10 above), so D quasi-embeds there; on the other hand, D does not embed in any product of two curves since it is contractible but not collapsible ([24]; see Remark 1.11 above).

As observed in [27], similar arguments show that the Cartesian power D^k quasi-embeds in a product of $2k$ trees, but does not embed in any product of $2k$ curves. (This uses the more general result of [24] that no polyhedron P with $\text{rk } H^1(P) < n$ and $H^n(P, P \setminus \{x\}) \neq 0$ for each $x \in P$ embeds in a product of n curves.)

Remark 1.19. Zeeman showed that $D \times I$ is collapsible [53], where D is the dunce hat. Hence $D \times I$ embeds in a product of 3 trees by Theorem 1.8. So the absolute retract X in Theorem 1.17 cannot be replaced by D . Moreover, it cannot be replaced by *any* 2-polyhedron R , since $R \times I$ embeds in a product of 3 trees by Proposition 1.3.

Conjecture 1.20. (a) *If a compact n -polyhedron P quasi-embeds in a product of n dendrites, then $P \times I$ embeds in a product of $n + 1$ trees.*

(b) *Same if P is a co-locally contractible (see §5) n -dimensional compactum.*

Theorem 1.17 should be compared with the following results.

Theorem 1.21 (Melikhov–Shchepin [36]). (a) *If X is a compact n -dimensional ANR that quasi-embeds in I^{2n-1} , $n > 3$, then $X \times I$ embeds in I^{2n} .*

(b) *If X is an acyclic n -dimensional compactum, $m > \frac{3(n+1)}{2}$ and $k > 0$, then the following are equivalent: (i) X embeds in I^m ; (ii) $X \times I^k$ embeds in I^{m+k} ; (iii) $X \times T^k$ embeds in I^{m+2k} , where T denotes the triod.*

In conclusion we note that the proof of non-embeddability in Theorem 1.17 involves the same kinds of local geometry and local algebra as that in the following

Theorem 1.22 (Melikhov–Shchepin [36]). *For each $n > 1$ there exists a compact n -dimensional ANR, quasi-embeddable but not embeddable in I^{2n} .*

2. COLLAPSIBLE POLYHEDRA

We use the following combinatorial notation [35; Chapter 2]. Given a poset P and a $\sigma \in P$, the *cone* $[\sigma]$ is the subposet of all $\tau \in P$ such that $\tau \leq \sigma$, and the *dual cone* $[\sigma]$ is the subposet of all $\tau \in P$ such that $\tau \geq \sigma$. The *link* $\text{lk}(\sigma, P)$ is the subposet of all $\tau \in P$ such that $\tau > \sigma$, and the *star* $\text{st}(\sigma, P)$ is the subposet of all $\rho \in P$ such that $\rho \leq \tau$ for some $\tau \in [\sigma]$. If K is a simplicial complex (viewed as a poset of nonempty faces ordered by inclusion), and $\sigma \in K$, then $\text{lk}(\sigma, K)$ is a simplicial complex, and $\text{st}(\sigma, K)$ is isomorphic to $[\sigma] * \text{lk}(\sigma, K)$.²

²Our $\text{lk}(\sigma, P)$ is a standard notion of link in modern Topological Combinatorics; we shall need it when P is a cubical complex (where every cone is isomorphic to the poset of nonempty faces of a cube). The

Here the join is defined as follows. The *dual cone* C^*P of the poset P consists of P together with an additional element $\hat{0}$ that is set to be less than every element of P . The *coboundary* ∂^*Q of a dual cone $Q = C^*P$, is the original poset P . (Note the relation with coboundary of cochains.) The *product* $P \times Q$ of two posets consists of pairs (p, q) , where $p \in P$ and $q \in Q$, ordered by $(p, q) \leq (p', q')$ if $p \leq q$ and $p' \leq q'$. The *join* $P * Q = \partial^*(C^*P \times C^*Q)$. Note that $P * Q = C^*P \times Q \cup P \times C^*Q$ (union along $P \times Q$).

The *canonical subdivision* $P^\#$ is the poset of all order intervals of P , ordered by inclusion. If K is a simplicial complex, then $(C^*K)^\#$ is a cubical complex. Conversely, if Q is a cubical complex and $q \in Q$, then $\text{lk}(q, Q)$ is a simplicial complex, and $\text{st}(q, Q)$ is isomorphic to $[q] \times (C^*\text{lk}(q, Q))^\#$. Moreover, $\text{lk}((p, q), P \times Q)$ is isomorphic to $\text{lk}(p, P) * \text{lk}(q, Q)$. The details can be found in [35; Chapter 2].

Theorem 1.8 now follows from

Lemma 2.1. *Let $K \searrow L$ be a simplicial collapse of simplicial complexes and let T_1, \dots, T_n be trees, so that $T = T_1 \times \dots \times T_n$ is a cubical complex. Suppose that $f: |L| \rightarrow |T|$ is a PL embedding such that $f(|\sigma|)$ is cubulated by a subcomplex of T for every simplex σ of L . Then each T_i embeds in a larger tree \tilde{T}_i and f extends to a PL embedding $\bar{f}: |K| \rightarrow |\tilde{T}|$, where $\tilde{T} = \tilde{T}_1 \times \dots \times \tilde{T}_n$, such that $\bar{f}(|\sigma|)$ is cubulated by a subcomplex of \tilde{T} for every simplex σ of K .*

Moreover, $|T| \cap \bar{f}(|K|) = f(|L|)$, and $|\tilde{T}|$ collapses onto $|T| \cup \bar{f}(|K|)$.

Proof. Arguing by induction, we may assume that $K \searrow L$ is an elementary simplicial collapse. Let Q denote the subcomplex of T cubulating $f(|L|)$, and let B be the subcomplex of Q cubulating the image of the topological frontier of $|L|$ in $|K|$. We may now forget K, L and f , remembering only that $|B|$ is a PL ball of some dimension $k < n$. We thus want to construct trees $\tilde{T}_i \supset T_i$ and a subcomplex β of $\tilde{T}_1 \times \dots \times \tilde{T}_n$ such that $\beta \cap Q = B$ and $|\beta|$ is a PL $(k+1)$ -ball.

The boundary of $|B|$ is cubulated by a subcomplex ∂B of B . Given a face $q = q_1 \times \dots \times q_n$ of $B \setminus \partial B$, we have $\text{lk}(q, T) \simeq \text{lk}(q_1, T_1) * \dots * \text{lk}(q_n, T_n)$. Each q_i is either a vertex or an edge, and then $\text{lk}(q_i, T_i)$ is either a finite set or the empty set, accordingly. Let C be set of those i for which q_i is a vertex. Then the cube $[q]$ is of dimension $n - \#C$, and consequently the dimension $d - 1$ of $\text{lk}(q, B)$ equals $k - n + \#C - 1 < \#C - 1$.

Every vertex v of $\text{lk}(q, T)$ lies in $\text{lk}(q_i, T_i)$ for some $i \in C$; in that case let us color v by the i th color. In particular, the subcomplexes $\Lambda := \text{lk}(q, Q)$ and $S := \text{lk}(q, B)$ of $\text{lk}(q, T)$ are C -colored. Since $\#C > d$, by the Fisk–Izmestiev–Witte lemma, the C -colored combinatorial $(d-1)$ -sphere S bounds (abstractly) a C -colored combinatorial ball D . Let Λ^+ be the amalgamated union $\Lambda \cup_S D$, that is, the pushout of the diagram $\Lambda \supset S \subset D$ in the category of C -colored simplicial complexes and color-preserving simplicial maps.

If $D \setminus S$ contains k_i vertices of color i , we define a new tree $T_i^+ = T_i \cup (q_i * [k_i])$ by attaching k_i new edges to T_i at the vertex q_i for each $i \in C$ (note that $[k_i] = \emptyset$ and so

notion of link in Combinatorial Topology of 1960s was something slightly different: being defined only when P is a simplicial complex, it is canonically *isomorphic* to our $\text{lk}(\sigma, P)$ but is not *identical* with it.

$T_i^+ = T_i$ for each $i \notin C$). Let $T^+ = T_1^+ \times \dots \times T_n^+$. The C -coloring of the vertices of $\text{lk}(q, T)$ extends to the similarly defined C -coloring of the vertices of $\text{lk}(q, T^+)$. Then any color-preserving identification of the vertices of $D \setminus S$ with the vertices of $\text{lk}(q, T^+)$ that are not in $\text{lk}(q, T)$ extends uniquely to a color-preserving simplicial map $\Lambda^+ \rightarrow \text{lk}(q, T^+)$ that extends the inclusion $\Lambda \subset \text{lk}(q, T)$. This simplicial map is injective on vertices, hence is an embedding. By construction, $\Lambda^+ \cap \text{lk}(q, T) = \Lambda$.

In particular, D is now identified with a subcomplex of $\text{lk}(q, T)$, hence $E := [q] \times (C^*D)^\#$ and $F := [q] \times D^\# \cup (\partial[q]) \times (C^*D)^\#$ are identified with subcomplexes of $[q] \times (C^*\text{lk}(q, T^+))^\# = \text{st}(q, T^+)$. Since $D \cap \Lambda = S$, we have $E \cap Q = E \cap B$. Let $Q^+ = Q \cup E$. Note that $E \cap B$ is the cubical combinatorial k -ball $\text{st}(q, B) = [q] \times (C^*\text{lk}(q, B))^\#$, and $F \cap B$ is its boundary, the cubical combinatorial sphere $\partial \text{st}(q, B) = [q] \times \text{lk}(q, B)^\# \cup (\partial[q]) \times (C^*\text{lk}(q, B))^\#$. Further note that $\text{st}(q, B) \setminus \partial \text{st}(q, B)$ is the dual cone $[q]$ of q in B . Then $B^+ = (B \setminus [q]) \cup F$ is a cubical combinatorial k -ball, which does not contain q .

Since $\Lambda^+ \cap \text{lk}(q, T) = \Lambda$, we have $Q^+ \cap T = Q$. Furthermore, $|T^+|$ collapses onto $|T \cup (q_1 * [k_1]) \times \dots \times (q_n * [k_n])|$ (using conewise collapses of the form $X \times CY \searrow X \cup Z \times CY$ where Z is a closed subpolyhedron of X), which in turn collapses onto $|T \cup E| = |T \cup Q^+|$ (using the collapse of the cone $|\prod_{i \in C} q_i * [k_i]|$ onto its subcone $|(C^*D)^\#|$).

In order to fit the above process in an inductive argument, let us now write Q_0, B_0 for the given Q, B . Assuming that Q_i, B_i have been constructed, along with some distinct $q_1, \dots, q_i \in (B_0 \setminus \partial B_0) \setminus B_i$, we repeat the above process with $Q = Q_i$ and $B = B_i$, with one modification: q is now not an arbitrary face of $B_i \setminus \partial B_i$, but one that is also a face of the original $B_0 \setminus \partial B_0$. Since q is still required to be a face of B_i , our hypothesis entails that $q \notin \{q_1, \dots, q_i\}$. We set $Q_{i+1} = Q^+$, $B_{i+1} = B^+$, and $q_{i+1} = q$. Then $q_0, \dots, q_{i+1} \in (B_0 \setminus \partial B_0) \setminus B_{i+1}$, which completes the inductive step. Since $B_0 \setminus \partial B_0$ is finite, the number of steps is bounded. If the final step is r th, it is easy to see that $B_r \cap B_0 = \partial B_0 = \partial B_r$, and $B_0 \cup B_r$ bounds a cubical combinatorial $(k+1)$ -ball β (namely, β is the union of all the $(k+1)$ -balls of the form E) such that $\beta \cap Q_0 = B_0$ and $\beta \cup Q_0 = Q_r$. \square

Remark 2.2. The combinatorial type of the ball β depends on the order in which q_1, \dots, q_r are picked out of $B_0 \setminus \partial B_0$. For instance, suppose that $n = 2$, $k = 1$ and the arc B_0 consists of e edges (and hence $e + 1$ vertex). If $e > 1$, then we may take q_1, \dots, q_r to be all the non-boundary vertices, ordered consecutively, which will lead to the same β as in [24]. For instance if $e = 2$ (so $r = 1$) and $T_1 = Q_0 = B_0$, $T_2 = pt$, then $\tilde{T}_1 = T_1$, \tilde{T}_2 is a single edge, and $Q_r = B_r = \tilde{T}_1 \times \tilde{T}_2$ (which amounts to two squares). On the other hand, if we first pick out all the edges (in any order) and then the $e - 1$ non-boundary vertices (in any order), the result will be unique, but quite different from the above. For instance if $e = 2$ (so $r = 3$) and $T_1 = Q_0 = B_0$, $T_2 = pt$, then at the final step \tilde{T}_1 is a triod, \tilde{T}_2 contains two edges, and B_r consists of four squares. Picking out only vertices but not consecutively may also lead to a β different from that in [24].

Remark 2.3. As discussed in the previous remark, the construction in the proof of Lemma 2.1 depends on the choices of the cubes q_1, \dots, q_r . Let us describe a canonical range of choices that all lead to the same embedding. Each tree T_i is constructed in stages $pt = T_{i0} \subset \dots \subset T_{is} = T_i$. The vertices of T_i are partially ordered by $v < w$ if there exists a $k < s$ such that $v \in T_{ik}$ and $w \notin T_{ik}$, yet w and v belong to the same component of $|T_i| \setminus |T_{i,k-1}|$. (In particular, incomparable vertices are non-adjacent in the tree.) This yields a partial order on the vertices of $B \setminus \partial B \subset Q \subset T_1 \times \dots \times T_n$. Let q_1, \dots, q_r be the vertices of $B \setminus \partial B$ arranged in some total order extending the constructed partial order. It is clear then that r is indeed the last stage of the construction, and that Q_r does not depend on the choice of the total order.

An alternative proof of the implication (iii) \Rightarrow (i) in Theorem 1.9 is given by Lemma 2.1 along with the following lemma (take $k = n - 1$).

Lemma 2.4. *Let L be an k -dimensional simplicial complex. Then there exist trees T_0, \dots, T_k and a PL embedding $f: |L| \rightarrow |T|$, where $T = T_0 \times \dots \times T_k$ such that $f(|\sigma|)$ is cubulated by a subcomplex of T for every simplex σ of L .*

The prejoin $P + Q$ consists of the elements of $P \cup Q$ with the order \preceq defined as follows: $p \preceq q$ iff either $p, q \in P$ and $p \leq q$ in P ; or $p, q \in Q$ and $p \leq q$ in Q ; or $p \in P$ and $q \in Q$. Note that $C^*P \simeq pt + P$. It is easy to see that $(P + Q)^b \simeq P^b + Q^b$, where P^b denotes the barycentric subdivision (see details in [35]).

Proof. Let S_i be the set of i -dimensional simplices of L . Then L is a subcomplex of $S_0 + \dots + S_k$. Hence L^b is a subcomplex of $(S_0 + \dots + S_k)^b \simeq S_0 * \dots * S_k$, which in turn is a subcomplex of $C^*(S_0 * \dots * S_k)$. Therefore $(L^b)^\#$ is a subcomplex of $(C^*(S_0 * \dots * S_k))^\# \simeq (C^*S_0)^\# \times \dots \times (C^*S_k)^\#$. Each $(C^*S_i)^\#$ is a tree, and the assertion follows. \square

3. LOCAL COHOMOLOGY

By H^* we denote the Alexander–Spanier cohomology [45], [31], or equivalently (see [44]) sheaf cohomology with constant coefficients [9]. If the coefficients are omitted, they are understood to be integer. The case of coefficients in a field is much easier (see [50]) but will not suffice for our purposes.

If (X, Y) is a pair of compacta, $H^i(X, Y)$ is isomorphic to the direct limit $\varinjlim H^i(P_j, Q_j)$, where $\dots \rightarrow (P_1, Q_1) \rightarrow (P_0, Q_0)$ is any inverse sequence of pairs of compact polyhedra with inverse limit (X, Y) . In particular, every cohomology group $H^i(Y, X)$ is countable.

More generally, when Y is closed in X (which we always assume to be metrizable), then $H^i(X, Y)$ coincides (see [44]) with the Čech cohomology of (X, Y) , which may be defined as the direct limit of the i th cohomology groups of the nerves of all open coverings of (X, Y) . In particular, if Y is closed in X and X is n -dimensional, then $H^i(X, Y) = 0$ for $i > n$ (since covers with at most n -dimensional nerve form a cofinal subset in the directed set of all open covers of X).

If X is a compactum and $x \in X$, the local cohomology group $H^i(X, X \setminus \{x\})$ is isomorphic to $\varinjlim H^{i-1}(U_i \setminus \{x\})$, where $U_0 \supset U_1 \supset \dots$ are neighborhoods of x in X .

such that $\bigcap U_k = \{x\}$ and each $\text{Int } U_k \supset \text{Cl } U_{k+1}$. As observed in [43; §1], this follows from the exact sequences of the pairs $(U_k, U_k \setminus \{x\})$ and the fact that the direct limit functor preserves exactness of sequences. However, this isomorphism will not be used in the sequel.

Instead, we shall use the following more geometric description of the local cohomology groups (parallel to [38; proof of Lemma 1]).

Proposition 3.1. *Let X be a compactum, let $x \in X$ and let $U_1 \supset U_2 \supset \dots$ be neighborhoods of x in X such that $\bigcap U_k = \{x\}$ and each $\text{Int } U_k \supset \text{Cl } U_{k+1}$. Then*

$$H^i(X, X \setminus \{x\}) \simeq H^i(X \times [0, \infty), X \times [0, \infty) \setminus U_{[0, \infty)}),$$

where $U_{[0, \infty)} = U_0 \times [0, 1) \cup U_1 \times [1, 2) \cup U_2 \times [2, 3) \cup \dots$.

Note that if the U_k are open, then $X \times [0, \infty) \setminus U_{[0, \infty)}$ is a closed subset of $X \times [0, \infty)$. Hence from the preceding discussion we obtain

Corollary 3.2. *If X is an n -dimensional compactum, $H^i(X, X \setminus \{x\}) = 0$ for $i > n + 1$ and all $x \in X$.*

Proof of Proposition 3.1. We shall show that $(X, X \setminus \{x\})$ is “almost” homotopy equivalent to the mapping telescope of pairs $(X, X \setminus U_i)$, meaning that there is a map of pairs in one direction, which admits a homotopy inverse separately on each entry of the pair; by the Five Lemma, this is just good enough as long as cohomology is concerned.

The projection $X \times [0, \infty) \rightarrow X$ yields a map of pairs $f: (X \times [0, \infty), X \times [0, \infty) \setminus U_{[0, \infty)}) \rightarrow (X, X \setminus \{x\})$. If $\varphi: X \setminus \{x\} \rightarrow [0, \infty)$ is a map such that $\varphi^{-1}([0, n]) \subset X \setminus U_n$, then $g: X \setminus \{x\} \rightarrow X \times [0, \infty)$ defined by $g(y) = (y, \varphi(y))$ is an embedding into $X \times [0, \infty) \setminus U_{[0, \infty)}$. It is easy to see that g is homotopy inverse to the restriction $h: X \times [0, \infty) \setminus U_{[0, \infty)} \rightarrow X \setminus \{x\}$ of the projection $X \times [0, \infty) \rightarrow X$; hence h is a homotopy equivalence. Using the isomorphisms induced by g and the homotopy equivalence $X \times [0, \infty) \rightarrow X$, the Five Lemma implies that f^* is an isomorphism. \square

By well-known arguments (see [37; proof of Theorem 4] or [34; proof of equation (*)] in §1.B or proof of Theorem 3.1(b)), Proposition 3.1 gives rise to a Milnor-type natural short exact sequence (found explicitly in [18]):

$$0 \rightarrow \varprojlim^1 H^{i-1}(X, X \setminus U_k) \rightarrow H^i(X, X \setminus \{x\}) \rightarrow \varprojlim H^i(X, X \setminus U_k) \rightarrow 0.$$

In particular,

$$H^{n+1}(X, X \setminus \{x\}) \simeq \varprojlim^1 H^n(X, X \setminus U_k), \quad (*)$$

if X is an n -dimensional compactum.

Lemma 3.3. *If X and Y are compacta of dimensions n and m , and $x \in X$ and $y \in Y$ are such that $H^{n+1}(X, X \setminus \{x\}) = 0$ and $H^{m+1}(Y, Y \setminus \{y\}) = 0$, then also $H^{n+m+1}(X \times Y, X \times Y \setminus \{(x, y)\}) = 0$.*

Proof. Since cohomology groups of pairs of compacta are countable, the hypothesis and the conclusion can be reformulated in terms of the Mittag-Leffler condition, using the

isomorphism $(*)$ and Gray's Lemma (see [34; Lemma 3.3]). Then the assertion follows (cf. [36; proof of Lemma 3.6(b)]) from the naturality in the Künneth formula [9; Theorem II.15.2 and Proposition II.12.3] (see also [31; Theorem 7.1], which implies the relative case using the map excision axiom). \square

Lemma 3.4. *If X is an n -dimensional compactum and $H^{n+1}(X, X \setminus \{x\}) = 0$, then $H^{n+1}(Y, Y \setminus \{x\}) = 0$ for every n -dimensional compactum $Y \subset X$ containing x .*

Proof. Let U_k be open neighborhoods of x in X as in Proposition 3.1. The restriction map $H^n(X, X \setminus U_k) \xrightarrow{f_k} H^n(Y, Y \setminus U_k)$ is onto from the exact sequence of the triple $(X, Y \cup (X \setminus U_k), X \setminus U_k)$, due to $H^{n+1}(X, Y \cup (X \setminus U_k)) = 0$. Then $\lim_{\leftarrow}^1 f_k$ is onto from the six-term exact sequence of inverse and derived limits (see [34; Theorem 3.1(d)] for a geometric proof) associated to the short exact sequences

$$0 \rightarrow \ker f_k \rightarrow H^n(X, X \setminus U_k) \xrightarrow{f_k} H^n(Y, Y \setminus U_k) \rightarrow 0.$$

But by naturality of the isomorphism $(*)$, $\lim_{\leftarrow}^1 f_k$ is identified with the restriction map $H^{n+1}(X, X \setminus \{x\}) \rightarrow H^{n+1}(Y, Y \setminus \{y\})$. \square

Remark 3.5. The Menger curve M contains points x such that $H^2(M, M \setminus \{x\}) \neq 0$. (Since M is known to be homogeneous, this applies to every $x \in M$.) For let Y be the subspace $\mathbb{N}^+ \times [0, 1) \cup [0, \infty] \times \{1\}$ of $[0, \infty] \times [0, 1]$, where $\mathbb{N}^+ = \{0, 1, \dots, \infty\}$, and let $y = (\infty, 1) \in Y$. Let us represent $Y \setminus \{y\}$ as a union $\bigcup K_i$, where each K_i is compact and lies in $\text{Int } K_{i+1}$. (And not just in K_{i+1} .) Then $\dots \rightarrow \tilde{H}^0(K_1) \rightarrow \tilde{H}^0(K_0)$ is of the form $\dots \rightarrow \bigoplus_{S_1} \mathbb{Z} \rightarrow \bigoplus_{S_0} \mathbb{Z}$, where $S_0 \supset S_1 \supset \dots$ is a nested sequence of infinite countable sets with $\bigcap S_i = \emptyset$. Since $H^1(Y) = 0 = \tilde{H}^0(Y)$, the inverse sequence $\dots \rightarrow H^1(Y, K_1) \rightarrow H^1(Y, K_0)$ is of the same form. Clearly it does not satisfy the Mittag-Leffler condition and consists of countable groups, so by Gray's Lemma (see [34; Lemma 3.3]) its derived limit is nontrivial. (In fact, it is easy to see, similarly to [34; Example 3.2], that $\lim_{\leftarrow}^1 H^1(Y, K_i) \simeq \prod_{\mathbb{N}} \mathbb{Z} / \bigoplus_{\mathbb{N}} \mathbb{Z}$.) Thus by $(*)$, $H^2(Y, Y \setminus \{y\}) \neq 0$. Since Y embeds into M , Lemma 3.4 implies $H^2(M, M \setminus \{x\}) \neq 0$, where x is the image of y .

Lemma 3.6. *If X is a local dendrite, then $H^2(X, X \setminus \{x\}) = 0$ for every $x \in X$.*

The proof is a bit technical; let us explain informally some intuition behind it. There are just two basic examples of inverse sequences of countable abelian groups with nonzero \lim_{\leftarrow}^1 : (i) $\dots \xrightarrow{p_1} \mathbb{Z} \xrightarrow{p_0} \mathbb{Z}$, each p_i being a nonzero prime (this occurs in the Skliarienko compactum), and (ii) $\dots \hookrightarrow \bigoplus_{i=1}^{\infty} \mathbb{Z} \hookrightarrow \bigoplus_{i=0}^{\infty} \mathbb{Z}$ (this occurs in Remark 3.5 and is called “Jacob's ladder” in [20]). Example (i) cannot occur in $(*)$ with $n = 1$, because there is “not enough room for twisting” in one-dimensional spaces, so we cannot expect to find even a single multiplication as in (i). On the other hand, if X is an LC_n compactum, then we cannot find example (ii) in $(*)$, because n -cohomology of compact subsets of X is “almost” finitely generated in the sense that for every two compact subsets $K \subset X$ and $L \subset \text{Int } K$, the image of $H^n(K) \rightarrow H^n(L)$ is finitely generated [9; II.17.5 and V.12.8].

Proof. Let us represent $X \setminus \{x\}$ as a union $\bigcup K_i$, where each K_i is compact and lies in $\text{Int } K_{i+1}$. Since X is locally contractible, for each n (in particular, for $n = 1$), each inclusion map $K_i \rightarrow K_{i+1}$ factors through a (not necessarily embedded in X) LC_n compactum L_i [34; Theorem 6.11]. We recall that LC_n compacta have finitely generated cohomology and (Steenrod) homology in dimensions $\leq n$ (see [9; II.17.7 and V.12.8], [34; 6.11]). Universal coefficients formulas then imply that LC_1 compacta have free abelian H^1 (see [9; V.12.8]) and consequently also free abelian H_0 (see [9; §V.3, Eq. (9) on p. 292]).

Consider a composition $f: L_i \rightarrow K_{i+1} \rightarrow K_j \rightarrow L_j$. By the naturality of the universal coefficients formula (see [9; V.12.8, V.13.7]), $f^*: H^0(L_j) \rightarrow H^0(L_i)$ is dual to $f_*: H_0(L_i) \rightarrow H_0(L_j)$. The image of f_* is a subgroup of the free abelian group $H_0(L_j)$. So it is itself free abelian, in particular, projective as a \mathbb{Z} -module. Hence f_* is a split epimorphism onto its image. Then the inclusion of the image of f_* into $H^0(L_i)$ is a split monomorphism. (Indeed, given abelian group homomorphisms $f_*: G \rightarrow H$, $f^*: \text{Hom}(H, \mathbb{Z}) \rightarrow \text{Hom}(G, \mathbb{Z})$ defined by $f^*(\psi) = \psi f_*$, and $s: \text{im } f_* \rightarrow G$ such that $f_* s f_* = f_*$, define $r: \text{Hom}(G, \mathbb{Z}) \rightarrow \text{im } f^*$ by $r(\varphi) = \varphi s f_*$; then $r f^* = f^*$, i.e. $r(\psi f_*) = \psi f_*$ for each $\psi \in \text{Hom}(H, \mathbb{Z})$.) Thus f^* is a homomorphism onto a direct summand of $H^0(L_i)$. The finitely generated group $H^0(L_i)$ contains no infinitely decreasing chain of direct summands; so the inverse sequence $\cdots \rightarrow H^0(L_1) \rightarrow H^0(L_0)$ satisfies the Mittag-Leffler condition. Hence so does $\cdots \rightarrow H^0(K_1) \rightarrow H^0(K_0)$.

On the other hand, consider a composition $g: L_i \rightarrow K_{i+1} \rightarrow X$. The image of $g^*: H^1(X) \rightarrow H^1(L_i)$ is a subgroup of the free abelian group $H^1(L_i)$. So it is itself free abelian, in particular, projective as a \mathbb{Z} -module. Hence g^* is a split epimorphism onto its image. Then the kernel of g^* is a direct summand in $H^1(X)$. The finitely generated group $H^1(X)$ contains no infinitely decreasing chain of direct summands; hence the homomorphisms $H^1(X) \rightarrow H^1(L_i)$ have the same kernel for all sufficiently large i . Then so do the homomorphisms $H^1(X) \rightarrow H^1(K_i)$. Since X is 1-dimensional, the latter are surjective. Hence $H^1(K_{i+1}) \rightarrow H^1(K_i)$ are isomorphisms for sufficiently large i . In particular, $\cdots \rightarrow H^1(K_1) \rightarrow H^1(K_0)$ satisfies the dual Mittag-Leffler condition.

Thus by Dydak's Lemma (see [34; Lemma 3.11]), $\cdots \rightarrow H^1(X, K_1) \rightarrow H^1(X, K_0)$ satisfies the Mittag-Leffler condition. Hence $\lim_{\leftarrow} {}^1H^1(X, K_i) = 0$, and the assertion follows from (*). \square

4. SKLIARIENKO'S COMPACTUM

We note that if the compactum X is the limit of an inverse sequence of compacta X_i , all of which embed in Y , then X quasi-embeds in Y (for it follows from the definition of the topology of the inverse limit that the maps $X \xrightarrow{p_i^\infty} X_i \subset Y$ are ε_i -maps with respect to any fixed metric on X , where $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$). The converse implication (which we shall not need here) holds when Y is a polyhedron (a simple proof should appear in a future version of [36]; see also [32; Theorem 1] but beware that their “ ε -maps” are required to be surjective).

4.1. Skliarienko's compactum. Given a direct sequence $X_1 \rightarrow X_2 \rightarrow \dots$, the *mapping telescope* $\text{Tel}(X_1 \rightarrow X_2 \rightarrow \dots)$ is the infinite union $MC(X_1 \rightarrow X_2) \cup_{X_2} MC(X_2 \rightarrow X_3) \cup_{X_3} \dots$ of the mapping cylinders (the direct limit of the finite unions). Let X be the one-point compactification of the mapping telescope of the direct sequence

$$S^1 \xrightarrow{2} S^1 \xrightarrow{2} \dots$$

of two-fold coverings. It is easy to see that X is a contractible and locally contractible 2-dimensional compactum, and so an AR. It was introduced by Je. G. Skliarienko [43; Example 4.6]. We shall call X the *Skliarienko compactum*.

Proposition 4.2. *Skliarienko's compactum quasi-embeds in a product of two dendrites.*

Proof. Let us represent X as an inverse limit of polyhedra. To this end, consider the following mapping telescope of a direct sequence:

$$X_i = \text{Tel}(S_1^1 \xrightarrow{2} \dots \xrightarrow{2} S_i^1 \rightarrow pt),$$

where each S_j^1 stands for a copy of S^1 . Note that X contains the cone $D^2 = \text{Tel}(S_i^1 \rightarrow pt)$. Let $f_i: X_{i+1} \rightarrow X_i$ be the composition of the quotient map $X_{i+1} \rightarrow X_{i+1}/D^2$ and a homeomorphism $X_{i+1}/D^2 \rightarrow X_i$ which is the identity on $\text{Tel}(S_1^1 \xrightarrow{2} \dots \xrightarrow{2} S_i^1)$. Then X is homeomorphic to the inverse limit of $\dots \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1$.

Notice that each X_i is a collapsible 2-polyhedron. Hence by a result of Koyama, Krasinkiewicz and Spieź (see Corollary 1.10), X_i embeds in a product of two trees T_i and T'_i . Let us consider the cluster $T = \lim_{\leftarrow} (\dots \rightarrow T_1 \vee T_2 \vee T_3 \rightarrow T_1 \vee T_2 \rightarrow T_1)$ of the T_i , where the basepoint of each T_i is one of its endpoints. Let T' be the analogous cluster of the trees T'_i . Then T and T' are dendrites, T contains a copy of each T_i , and T' contains a copy of each T'_i . Thus each X_i embeds in $T \times T'$. Therefore X quasi-embeds there. \square

Let X be the Skliarienko compactum and let $\infty \in X$ denote the remainder point of the one-point compactification. It is easy to see that $H^3(X, X \setminus \{\infty\})$ is non-zero [43]. More generally, let us compute $H^{3+k}(X \times I^k, X \times I^k \setminus \{(\infty, 0)\})$, where $I = [-1, 1]$. Let F_i be the union of the first i mapping cylinders in the mapping telescope:

$$F_i = \text{Tel}(S_1^1 \xrightarrow{2} \dots \xrightarrow{2} S_i^1).$$

Each F_i collapses onto S_i^1 , and these collapses identify up to homotopy the inclusions $F_i \subset F_{i+1}$ with the two-fold coverings $S_i^1 \xrightarrow{2} S_{i+1}^1$. Hence the inverse sequence $\dots \rightarrow H^1(F_2) \rightarrow H^1(F_1)$ is of the form $\dots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$. Since X is an AR, so is the inverse sequence $\dots \rightarrow H^2(X, F_2) \rightarrow H^2(X, F_1)$. Let $G_i = F_i \times I^k \cup X \times (I^k \setminus (-\frac{1}{i}, \frac{1}{i})^k)$. By the Künneth formula (see references in the proof of Lemma 3.3), $H^{2+k}(X \times I^k, G_i) \simeq H^2(X, F_i)$, and the inverse sequence $\dots \rightarrow H^{2+k}(X \times I^k, G_2) \rightarrow H^{2+k}(X \times I^k, G_1)$ is again of the form $\dots \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z}$. In particular, it does not satisfy the Mittag-Leffler condition, so by Gray's Lemma (see [34; Lemma 3.3]) its derived limit is nontrivial. (In

fact, it is easy to compute that it is isomorphic to \mathbb{Z}_2/\mathbb{Z} , where \mathbb{Z}_2 is the group of 2-adic integers; see [34; Example 3.2].) Thus by (*), $H^{3+k}(X \times I^k, X \times I^k \setminus \{(\infty, 0)\}) \neq 0$.

Theorem 4.3. *If X is the Skliarienko's compactum, $X \times I^k$ does not embed in any product of $2 + k$ local dendrites.*

Proof. Suppose $X \times I^k \subset Y_1 \times \dots \times Y_n$, where Y_i are local dendrites. Then $(\infty, 0) \in X \times I^k$ is of the form (y_1, \dots, y_n) . By Lemma 3.6, $H^2(Y_i, Y_i \setminus \{y_i\}) = 0$ for each i . Then by Lemma 3.3, $H^{3+k}(\prod Y_i, \prod Y_i \setminus \{(y_i)\}) = 0$. Therefore by Lemma 3.4, $H^{3+k}(X \times I^k, X \times I^k \setminus \{(\infty, 0)\}) = 0$. This contradicts the above computation. \square

Theorem 4.4 (Koyama–Krasinkiewicz–Spieź). *If a compact n -dimensional ANR embeds in a product of n curves, then it embeds in a product of n local dendrites.*

Proof. It is well-known that locally contractible compacta have finitely generated cohomology groups (see [9; II.17.7], [34; 6.11]). If a locally connected n -dimensional compactum X with finitely generated $H^n(X)$ embeds in a product n curves, then the first several lines of the proof of Theorem 2.B.1 in [24] (which contain further references) produce an embedding of X in a product of n local dendrites. \square

Theorems 4.3 and 4.4 have the following

Corollary 4.5. *Skliarienko's compactum multiplied by I^k does not embed in any product of $2 + k$ curves.*

Corollary 4.5 combines with Proposition 4.2 to imply Theorem 1.17.

Remark 4.6. If $\dots \rightarrow G_1 \rightarrow G_0$ is an inverse sequence of countable groups, let $\varprojlim^1_{fg} G_i$ be the direct limit $\varinjlim L_\alpha$ of the derived limits $L_\alpha = \varprojlim^1 H_{\alpha i}$ over all inverse sequences $\dots \rightarrow H_{\alpha 1} \rightarrow H_{\alpha 0}$ of finitely generated subgroups $H_{\alpha i} \subset G_i$, where the bonding maps are the restrictions of those in $\dots \rightarrow G_1 \rightarrow G_0$. Some results about \varprojlim^1_{fg} will appear in a future paper by the first author. By using the functor \varprojlim^1_{fg} in place of \varprojlim^1 , it should be possible to refine the proof of Theorem 4.3 so as to obtain a purely algebraic proof of Corollary 4.5, without using Theorem 4.4.

Remark 4.7. The same arguments (only using the general case of Theorem 1.8 rather than the easier 2-dimensional case) show that the n -dimensional Skliarienko compactum (similarly defined with S^{n-1} in place of S^1) quasi-embeds in a product of n dendrites, but does not embed in a product of n curves.

5. CO-LOCAL CONTRACTIBILITY

Let us call a compactum X *co-locally contractible* at $x \in X$ if every neighborhood U of x contains a neighborhood V of x such that the inclusion $X \setminus \{x\} \subset X$ is homotopic to a map $X \setminus \{x\} \rightarrow X \setminus V \subset X$ by a homotopy keeping $X \setminus U$ fixed. (Equivalently, every neighborhood U of x contains a neighborhood V of x such that for every neighborhood W of x contained in V , the inclusion $X \setminus W \subset X$ is homotopic to a map $X \setminus W \rightarrow X \setminus V$

by a homotopy keeping $X \setminus U$ fixed.) We call X *co-locally contractible* if it is co-locally contractible at every point. (Compare Borsuk's idea of colocalization [5; §IX.16] and colocal connectedness of Krasinkiewicz and Minc [26].)

Remark 5.1. A slightly stronger property than co-local contractibility, obtained by replacing the inclusion $X \setminus \{x\} \subset X$ with the identity map of $X \setminus \{x\}$, is known as *reverse* (or *backward*) *tameness* of $X \setminus \{x\}$ (see [41], [20]). Dually, $X \setminus \{x\}$ is called *forward tame* if there exists a closed neighborhood U of x such that for every neighborhood V of x , the inclusion $V \setminus \{x\} \subset X \setminus \{x\}$ is properly homotopic to a map $V \setminus \{x\} \rightarrow U \setminus \{x\} \subset X \setminus \{x\}$ (see [41], [20]). It is not hard to see (even if appears surprising) that forward tameness of $X \setminus \{x\}$ implies local contractibility of X at x . To see that the converse implication fails, let P be the suspension of a non-contractible acyclic polyhedron and let its basepoint b be one of the two suspension points; or alternatively let P be the dunce hat and b its unique 0-cell. Then the cluster $C = \lim (\cdots \rightarrow P \vee P \vee P \rightarrow P \vee P \rightarrow P)$ of copies of P is an AR, yet it follows from Dydak–Segal–Spież [14] that $C \setminus \{b\}$ is not forward tame.

Proposition 5.2. *If an n -dimensional compactum X is co-locally contractible at x , then $H^{n+1}(X, X \setminus \{x\}) = 0$.*

Proof. This is a straightforward diagram chasing. The hypothesis implies that, with x , U and V as above and for each i , the restriction map $H^i(X \setminus \{x\}) \rightarrow H^i(X \setminus V)$ is a split injection on the image of $H^i(X)$. Hence the image of the forgetful map $f: H^i(X \setminus \{x\}, X \setminus V) \rightarrow H^i(X \setminus \{x\})$ lies in the image of $H^i(X)$. The latter equals the kernel of the coboundary map $\delta: H^i(X \setminus x) \rightarrow H^{i+1}(X, X \setminus \{x\})$, hence $\delta f = 0$. Since this $\delta f: H^i(X \setminus \{x\}, X \setminus V) \rightarrow H^{i+1}(X, X \setminus \{x\})$ is also the coboundary map, the restriction $H^{i+1}(X, X \setminus \{x\}) \rightarrow H^{i+1}(X, X \setminus V)$ must be an injection. Finally, since X is n -dimensional and without loss of generality V is open, $H^{n+1}(X, X \setminus V) = 0$. Thus $H^{n+1}(X, X \setminus \{x\}) = 0$. \square

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